Math 210C Lecture 1 Notes

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1 Radicals and Primary Ideals

1.1 Radicals

We will be assuming that R be a commutative ring.

Definition 1.1. The **radical** of an ideal I of R is $\sqrt{I} = \{a \in R : a^k \in I \text{ for some } k \geq 1\}$.

Lemma 1.1. *$\sqrt{I}$ is an ideal.*

Proof. If $r \in R$ and $a \in \sqrt{I}$, let k be such that $a^k \in I$. Then $(ra)^k = r^k a^k \in I$, so $ra \in \sqrt{I}$. If $a, b, \in \sqrt{I}$, then $a^k, b^\ell \in I$. Then $(a+b)^{k+\ell} = \sum_{i=0}^{k+\ell} \binom{k+\ell}{i} a^i b^{k+\ell-i}$. Either $i \geq k$ or $k + \ell - i \geq \ell$, so $(a+b)^{k+\ell} \in I$. So $a + b \in \sqrt{I}$. $\square$

Definition 1.2. The **nilradical** $\sqrt{0} = \{a \in R : a \text{ is nilpotent}\}$ is the radical of 0.

Example 1.1. Let $R = F[x]/(x^n)$, where F is a field. Then $\sqrt{0} = (x)$.

Lemma 1.2. *If $\pi : R \to R/I$ is a projection, then $\pi(\sqrt{I})$ is the nilradical of R/I.*

Proposition 1.1. *Let I be a proper ideal of R. Then*

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ I \subseteq \mathfrak{p}}} \mathfrak{p}.$$

Proof. One direction uses Zorn's lemma. $\square$

Definition 1.3. A ideal is **radical** if $I = \sqrt{I}$.[1]

Example 1.2. Prime ideals are radical. If $a^n = aa^{n-1} \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $a^{n-1} \in \mathfrak{p}$. By recursion, $a \in \mathfrak{p}$.

[1] An ideal is also called radical if it's just really really cool.

Example 1.3. Let F be a field, let $f_1, \ldots, f_r$ be irreducible in $F[x]$, and let $k_1, \ldots, k_r \geq 1$. Then $\sqrt{(f_1^{k_1}, \ldots, f_r^{k_r})} = (f_1, \ldots, f_r)$.

Example 1.4. If I is an ideal, $\sqrt{I}$ is radical. That is, $\sqrt{\sqrt{I}} = \sqrt{I}$.

Proposition 1.2. *Let R be noetherian and $I \subseteq R$ an ideal. There exists $N \geq 1$ such that $(\sqrt{I})^N \subseteq I$.*

Proof. In a Noetherian ring, all ideals are finitely generated: $\sqrt{I} = (a_1, \ldots, a_m)$. There exist $k_i \geq 1$ such that $a_i^{k_i} \in I$. Let $k = \max(k_i)$. Then $a_i^k \in I$ for all i. For arbitrary elements, let $x = \sum_{i=1}^m r_i a_i \in \sqrt{I}$, where $r_i \in R$. Then $x^{mk} \in (\{a_1^{i_1}, \ldots, a_m^{i_m} : i_j \geq 0, i_1 + \cdots + i_m = km\}) \subseteq (a_1^k, \ldots, a_m^k) \subseteq I$. $\square$

Example 1.5. This property does not need to hold in non-noetherian rings. Let $R = F[x_1, x_2, x_3, \ldots]/(x_1, x_2^2, x_3^3, \ldots)$. Then $\sqrt{0} = (x_1, x_2, x_3, \ldots)$. But $(\sqrt{0})^n \neq (0)$ for all $n \geq 1$.

Definition 1.4. $I \subseteq R$ is **nilpotent** if there exists $n \geq 1$ such that $I^n = (0)$.

Corollary 1.1. *In a noetherian ring, the nilradical is nilpotent.*

1.2 Primary ideals

Primary ideals are a generalization of prime ideals.

Definition 1.5. A proper ideal $\mathfrak{q}$ of R is **primary** if for any $a, b \in R$ such that $ab \in \mathfrak{q}$, either $a \in \mathfrak{q}$ or $b^n \in \mathfrak{q}$ for some $n \geq 1$.

Remark 1.1. Since R is commutative, this condition is symmetric, even if it does not look so at first.

Lemma 1.3. *A proper ideal $\mathfrak{q}$ of R is primary if and only if every zero divisor in $R/\mathfrak{q}$ is nilpotent.*

Proposition 1.3. *The radical of any primary ideal is a prime ideal.*

Proof. Let $\mathfrak{q}$ be primary. Suppose $ab \in \sqrt{\mathfrak{q}}$. Then there exists $k \geq 1$ such that $a^k b^k \in \mathfrak{q}$. Then either $a^k \in \mathfrak{q}$ or $b^{nk} \in \mathfrak{q}$ for some $n \geq 1$. That is, either $a \in \sqrt{\mathfrak{q}}$ or $b \in \sqrt{\mathfrak{q}}$. $\square$

Definition 1.6. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, where $\mathfrak{q}$ is primary, we say $\mathfrak{q}$ is **$\mathfrak{p}$-primary**. We say $\mathfrak{p}$ is the **associated prime** of $\mathfrak{q}$.

Example 1.6. Let F be a field. Then $(x^2, y) \subseteq F[x, y]$ is primary: $F[x, y]/(x^2, y) \cong F[x]/(x^2)$, so by the lemma, (x^2, y) is primary with $\sqrt{(x^2, y)} = (x, y)$.

Example 1.7. Let $R = F[x,y,z]/(xy - x^2)$, and let $\mathfrak{p} = (x,z)$. Then $R/\mathfrak{p} \cong F[y]$, so $\mathfrak{p}$ is prime. We have $\sqrt{\mathfrak{p}^2} = \mathfrak{p}$, and $xy \in \mathfrak{p}^2$, but $x \notin \mathfrak{p}^2$ and $y \notin \sqrt{\mathfrak{p}^2} = \mathfrak{p}$. So $\mathfrak{p}^2$ is not primary.

Lemma 1.4. *If I is an ideal of R such that $\sqrt{I}$ is maximal, then I is primary. In particular, any power of a maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-primary.*

Proof. Let $\mathfrak{m} = \sqrt{I}$ be maximal. The image of $\mathfrak{m}$ in R/I is the nilradical of R/I. The nilradical of R/I is the intersection of all prime ideals of R/I, so the nilradical is the only prime ideal in R/I. So R/I is local, and every non nilpotent element is a unit. Every zero divisor is therefore nilpotent. By the earlier lemma, I is primary. $\square$

Lemma 1.5. *A finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.*

Definition 1.7. Let I be an ideal of R. A **primary decomposition** of I is a finite collection $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ of primary ideals such that $I = \bigcap_{i=1}^n \mathfrak{q}_i$. A primary decomposition $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ of I is **minimal** if for all i, $\mathfrak{q}_i \not\supseteq \bigcap_{\substack{j=1 \\ j \neq i}}^n$ and $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ for all $i \neq j$.

Proposition 1.4. *There always exists a minimal primary decomposition if there exists a primary decomposition*

Proof. Use the lemma. $\square$

Theorem 1.1. *If R is noetherian, every proper ideal of R has a primary decomposition.*

We will prove this next time.